Ground state nonuniversality in the random-field Ising model

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Two attractive and often used ideas, namely, universality and the concept of a zero-temperature fixed point, are violated in the infinite-range random-field Ising model. In the ground state we show that the exponents can depend *continuously* on the disorder and so are nonuniversal. However, we also show that *at finite temperature* the thermal order-parameter exponent 1/2 is restored so that temperature is a relevant variable. Broader implications of these results are discussed.

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Cooperative behavior in disordered systems can usually be concisely characterized using scaling theories. These scaling theories contain scaling exponents and it is particularly important and satisfying if these exponents are independent of the fine details of the model, that is they are in some sense ''universal.'' If this holds it allows the theorists to study the simplest or most convenient model in a class in order to find the scaling exponents and, more importantly, that experiments should show the same exponents as the theory even though they may look very different on short length scales. Universality has been spectacularly successful in the study of phase transitions as a function of temperature, culminating in the development of the renormalization group [1]. There the critical exponents, usually, only depend on the symmetry of the order parameter and the spatial dimension. Due to the fact that scaling theories also work in many disordered systems it is natural to try to extend the ideas that work so well for thermal phase transitions to the disorder case.

Universality in disordered systems assumes that the critical exponents should *not depend on the type of disorder*, provided the disorder distribution is short-range correlated and provided it is not too broad. Universality with respect to disorder has been confirmed in some systems, with notable examples being percolation [2] and the problem of a directed polymer in a random medium [3]. However universality in disordered systems is unproven in general and indeed it has been questioned in the spin glass problem where there appears to be qualitative difference between the behavior in the presence of Gaussian as compared to bimodal disorder [4]. More recently universality has even been questioned in the random-field Ising model [5,6], which is one of the simplest models of a disordered material. Moreover the experimental tests of the random-field Ising exponents rely on universality [7,8] as the experiments are carried out on *diluted antiferromagnets in a field* [9] that are expected to lie in the same universality class (these experiments are also plagued by kinetic effects due to the large barriers that exist in random magnets). We show that universality fails in the ground state of the infinite-range random-field Ising model, as the *exponents may vary continuously* with the type of disorder.

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However, we also show that universality is restored at any finite temperature in the sense that at finite temperature the order parameter exponent is always 1/2, when the transition is continuous. This implies that another important concept in disordered systems, the concept of a *zero-temperature fixed point*, is violated in this model. The origin of the concept of a zero-temperature fixed point is that disorder usually provides a stronger perturbation than thermal fluctuations (which in turn are usually a stronger perturbation than quantum fluctuations). Thus a study of the ground states of disordered systems can lead to scaling theories that are qualitatively correct at finite temperatures. This is particularly attractive since there now exist methods for finding the exact ground states of many quenched random systems [10]. In the vernacular of random systems it is often stated that there exists a zero-temperature fixed point that controls the behavior at finite temperature. In particular, in both the random-field Ising model [11–14] and in spin glasses [4], scaling theories are frequently based on the assumption of a zero-temperature fixed point. However, we show that the mean-field theory of the random-field Ising model *is not*, in general, controlled by a zero-temperature fixed point.

We first demonstrate that the critical exponents can take on a range of values in the ground state of the random-field Ising model. The Hamiltonian for this model is

$$\mathcal{H} = -J_0 \sum_{ij} S_i S_j - \sum_i h_i S_i = NE_{ex} + NE_f, \tag{1}$$

where the first sum is over all spin pairs and $J_0 = J/N$ where N is the number of sites in the lattice to ensure an extensive energy. When the distribution of random fields is narrow, the exchange term dominates and the system is a ferromagnet (in dimensions greater than or equal to 3), while when the random-field distribution is broad the random field dominates and the system becomes a paramagnet. We take distributions of random fields that have mean zero and width δh, and consider the key ratio $H = \delta h / J$ that measures the strength of the random field in comparison to the exchange.

Consider a spin subspace in which the magnetization m is fixed, i.e., $m = (n_+ - n_-)/N$, where n_+ is the number of up spins in the configuration and n_- is the number of down spins. It is easy to find the lowest energy state for fixed m. The exchange energy is given by

$$E_{ex}(m)=\frac{-J}{2N^2}(n_+^2+n_-^2-2n_+n_-)=\frac{-Jm^2}{2}. \qquad (2)$$

Due to the fact that the exchange is of infinite range, all configurations at fixed m have the same energy and so are combinatorially degenerate. The field term splits this degeneracy by choosing the configuration that has the smallest field energy. This is achieved by satisfying the largest random fields and leaving the smallest possible fields unsatisfied. If the distribution of random fields is $P(h)$ (which we assume to be symmetric about the origin), then in the large lattice limit, we have

$$E_f=-2\int_0^\infty dhhP(h)+2\int_0^{h_c(m)} dhhP(h), \qquad (3)$$

where the first term is the ideal field energy in which every spin is oriented in the direction of its local field, and the second term is the energy cost due to the fractions of fields that are unsatisfied. The fractions of fields that are unsatisfied is determined by the magnetization,

$$m=2\int_0^{h_c(m)} P(h)dh. \qquad (4)$$

The ground state is found by determining the value of m which minimizes the energy (1)–(3), given the constraint (4). Carrying out the variation yields

$$\frac{\partial(E_{ex}+E_f)}{\partial m}=-Jm+2h_c(m)P(h_c(m))\frac{\partial h_c(m)}{\partial m}. \qquad (5)$$

By taking a derivative of Eq. (4) with respect to m (using the chain rule) we find, $1=2P(h_c(m))\partial h_c(m)/\partial m$. Using this to remove $\partial h_c(m)/\partial m$ from the right-hand side of Eq. (5) and setting Eq. (5) to zero, we find that the cutoff field is related to the magnetization via $h_c(m)=Jm$. Subsitution of this into Eq. (4) yields the *ground-state mean-field equation*

$$m=2\int_0^{Jm} P(h)dh. \qquad (6)$$

This equation gives the magnetization values at which the energy is extremal. Note that $m=0$ (the paramagnet) is always an extremum, as expected. Since we are treating the case of symmetric random fields, we can restrict attention to the case where $0\leqslant m\leqslant 1$. To determine whether an extrema is a maximum or a minimum, we need to evaluate the curvature near the extremum,

$$\left.\frac{\partial^2(E_{ex}+E_f)}{\partial m^2}\right|_{m_s}=-J+\frac{1}{2P(h_c(m_s))}. \qquad (7)$$

Finally, in order to determine the ground state, we need to compare the free energies of the solutions to the mean-field equation (4) with the energy of the magnetized state i.e., $m=1$.

An elegant result due to Aharony [15] states that the nature of the finite temperature phase transition in the infinite-range random-field Ising model depends on the curvature of the disorder distribution at the origin. Bimodal distributions lead to a first order jump in the order parameter at low temperatures (and hence a tricritical point at finite temperature), while unimodal distributions exhibit continuous transitions with exponent $\beta=1/2$, as originally found by Schneider and Pytte [16] for the case of Gaussian disorder. However, we now show that the exponent $\beta=1/2$ is *not universal* in the ground state.

We show that β may change continuously with the disorder, by considering the distribution of random fields given by

$$P(h)=\frac{y+1}{2yH}\left[1-\left(\frac{|h|}{H}\right)^y\right] \quad -H\leqslant h\leqslant H, \qquad (8)$$

with $y\geqslant 0$. In the limit $y\to\infty$ $P(h)\to$(uniform) so that a first order behavior is expected, while if $y\to 2$ it looks like a Gaussian near the origin so we expect a continuous transition with $\beta=1/2$. In the following discussion, we take $J=1$, so that H has been normalized by J. The distribution (8) is the first two terms in the expansion of the stretched exponential, $\exp[-(|h|/H)^y]$ that has the same critical behavior as Eq. (8). However, Eq. (6) can be solved exactly for the case (8) to yield (in addition to $m=0$),

$$m_s=H(y(H_c-H))^{1/y} \quad \text{for} \quad 1\leqslant H\leqslant H_c, \qquad (9)$$

where the critical field is given by, $H_c=(y+1)/y$. The lower bound on H is due to the cutoff in Eq. (8). For $H<1$, the exchange always wins and the magnetization is $m=1$. From Eq. (7), the second derivative is

$$\left.\frac{\partial^2(E_{ex}+E_f)}{\partial m^2}\right|_{m=0}=-1+\frac{yH}{y+1}. \qquad (10)$$

Thus the curvature at zero magnetization changes from positive (a minimum) for $H>H_c$ to negative for $H<H_c$. It is also easy to show that the solution m_s is always a minimum. Evaluating the energies at the three solutions $m=0$, $m=m_s$, $m=1$ yields a behavior typified by Fig. 1(a). For $H>H_c$, the ground state has $m=0$ and the system is a paramagnet, for $1\leqslant H\leqslant H_c$, the ground state has $m=m_s$ and is magnetized, while for $H<1$, the magnetization saturates. This behavior is summarized in the phase diagram of Fig. 1(b). The critical exponent $\beta=1/y$ on this upper curve in this figure and is clearly *nonuniversal in the ground state*.

However, when the transition is continuous, Aharony [15] has demonstrated that at finite temperature $\beta=1/2$ based on the mean-field equation for the random-field Ising model,

$$m=\int_{-\infty}^{\infty} dhP(h)\tanh(m/T+h/T). \qquad (11)$$

We now reconcile the ground-state result (9) found above with the finite temperature behavior found from equation (11). Assuming that $P(h)$ is symmetric, equation (11) can be reduced to

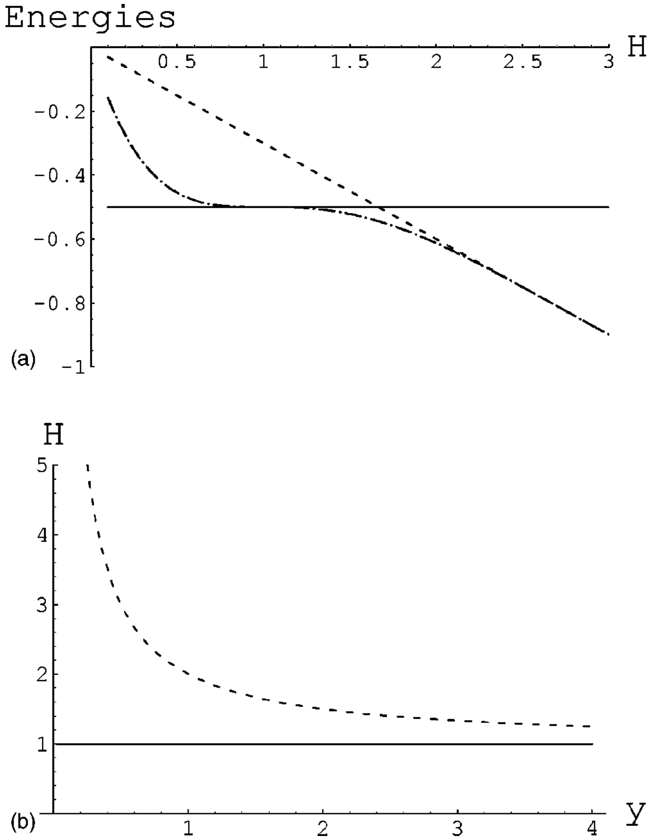

FIG. 1. (a) The ground-state energy as a function of the width of the magnetic field distribution H, for the case $y=1/2$. The flat curve is for $m=1$, the linearly decreasing curve is for $m=0$, while the third curve is for the solution m_s given in Eq. (9) of the text. (b) The upper curve is the dependence of the critical field on the exponent in the field distribution [i.e., $H=(y+1)/y$]. Above this line the magnetization is zero. The lower line is $H=1$, below which the magnetization is saturated (i.e. $m=1$ for $H<1$). Between these two lines the magnetization obeys Eq. (9) of the text, with the magnetization going to zero with exponent $1/y$ at the upper curve.

$$m=2\tanh(2m/T)\int_0^\infty \frac{P(h)dh}{1+\frac{\cosh(2h/T)}{\cosh(2m/T)}}. \qquad (12)$$

From this expression, it is seen that there are two regimes, $m/T\gg 1$ and $m/T\ll 1$. At zero temperature only the first regime holds, while at any finite temperature the second regime is applicable very close to the critical point.

When $m/T\gg 1$, $\tanh(m/T)\to 1$ and $\cosh(2h/T)/\cosh(2m/T)\to\exp[2(h-m)/T]$, which yields

$$m=2\int_0^\infty \frac{P(h)dh}{1+\exp[2(h-m)/T]} \quad m/T\to\infty. \qquad (13)$$

Now note that this expression looks like a Sommerfeld integral for the free Fermi gas, with the Fermi energy given by, $\epsilon_f=m$. The leading term at low temperatures is then the integral of $P(h)$ up to the Fermi energy, and hence is equivalent to the ground state result given in Eq. (6).

However, at any finite temperature, there is a regime in which the magnetization is small compared to the temperature, $m/T\ll 1$. In that case, $\cosh(2m/T)\to 1$, and Eq. (11) reduces to the mean-field theory for the thermal transition, but with a renormalized coefficient that depends on the field distribution, i.e.,

$$m=2I(H,T)\tanh(2m/T), \quad m/T\to 0, \qquad (14)$$

where

$$I(H,T)=\int_0^\infty \frac{P(h)dh}{1+\cosh(2h/T)}. \qquad (15)$$

Note that there is a factor of 2 difference in the argument of the tanh as compared to the thermal mean-field theory. However the critical temperature and critical exponent are the same. For any finite temperature, provided $m\ll T$, an expansion to third order in m of Eq. (14) shows that the magnetization approaches zero with exponent $\beta=1/2$. Moreover, Eq. (14) with (15) shows that the critical field and temperature are related to each other through the relation

$$T=4I(H_c(T),T), \qquad (16)$$

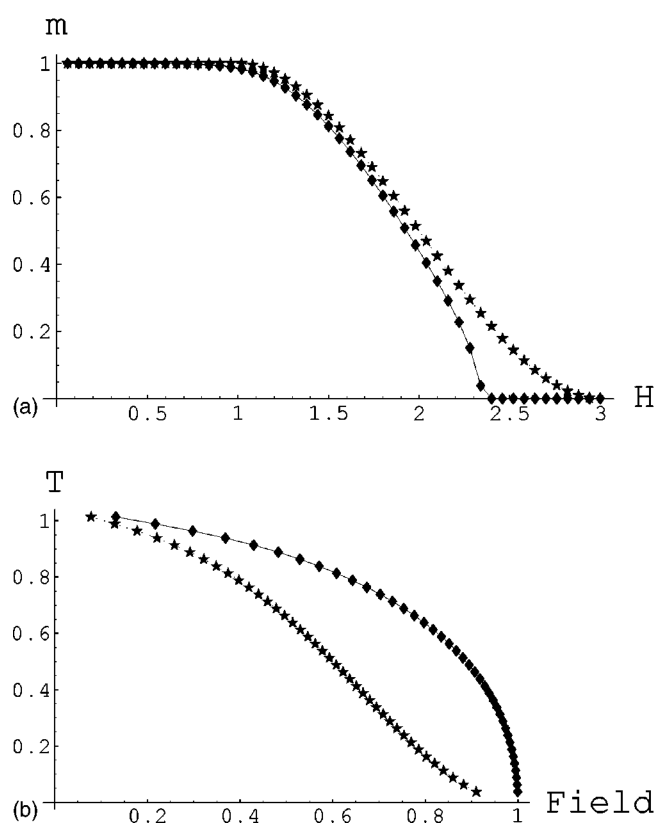

FIG. 2. (a) The magnetization as a function of the width of the random field (H) for $y=1/2$, for two temperatures. The upper curve is for zero temperature, while the lower curve is for $T=0.2$. (b) The H-T phase diagram [T vs $H_c(T)/H_c(0)$] for $y=2$ (upper curve) and $y=1/2$ (lower curve) found from solving Eq. (17) of the text.

provided the magnetization is continuous at the transition. For the probability distribution given in Eq. (8), this reduces to

$$\frac{H_c(T)}{H_c(0)} = \tanh\left[\frac{H_c(T)}{T}\right] - \left[\frac{T}{2H_c(T)}\right]^y \int_0^{2H_c(T)/T} \frac{x^y dx}{1+\cosh(x)}. \quad (17)$$

The magnetization as a function of field is given in Fig. 2(a) for $y=1/2$. From this figure it is seen that the critical exponent in the ground state is different from that at finite temperatures, and clearly illustrates the fact that temperature is a relevant variable. The temperature-field phase diagram is presented in Fig. 2(b) for the two cases $y=1/2$ and $y=2$. There is a sharp shift in the phase boundary with temperature for cases where y is small (rapidly decaying field distributions near the origin), which is strong indicator that temperature is relevant.

We have demonstrated the failure of universality in the ground state of the mean-field theory of the random-field Ising model. In addition the concept of a zero-temperature fixed point is invalid. The fact that the Gaussian distribution of random fields does have exponent $\beta=1/2$ in the ground state is atypical and should not be expected unless the disorder distribution is quadratic near the origin. Finite temperature introduces thermal fluctuations that are also Gaussian, which is the reason that the Gaussian distribution of disorder is special and atypical.

At first blush, our results raise serious questions about scaling theories of disordered systems based on a zero-temperature fixed point, and about the applicability of numerical studies in the ground state to finite temperature properties. However, this may not be the correct conclusion. Instead the conventional mean-field theory described here may be pathological and not typical of the behavior in finite dimensions. That itself would be a rather surprising result, which, for example, could be due to a renormalization of the disorder distribution to a Gaussian under rescaling in finite dimensions. These issues can only be resolved by careful studies of universality to disorder in finite dimensions, which is a difficult task except at zero temperature where exact numerical calculations are possible.

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